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INTEGRALS IN METEOROLOGY, HYDROLOGY AND IN GEOSCIENCE

Учебное пособие

ООО «Андреевский издательский дом»

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Пособие предназначено для студентов всех форм обучения по направлению подготовки 05.03.05 – Прикладная гидрометеорология и может быть полезно для студентов других специальностей.

Пособие предназначено для самостоятельной и индивидуальной работы студентов. Оно содержит краткий теоретический курс, примеры и описания решения задач с помощью табличного процессора EXCEL.

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1. Indefinite integral

Introduction.

In practical situations, we may be interested to know the position of an object at an instant, when velocity of the object at that instant is given. That is if s(t) is the displacement of an object in time (t) and we know $\frac{ds}{dt}$.

How can we find the displacement at time (t)? How do we find the velocity of a moving object, when its acceleration and initial velocity is known? These three problems involve the process of finding the function whose derivative is given. Integration and differentiation are a pair of inverse operations. So far, from a given function, we have been finding its derivative but the question arises: what is the function whose derivative is known? If the derivative of a function is given, then the function itself is called anti-derivative or integral. For example:

Consider the function f(x) = x^4 then its derivative is given by f '(x) = $4x^3$. The question arises: given f '(x) = $4x^3$ what is f(x)?

1.1. - Indefinite Integrals as the Anti-derivative.

Consider the following example: Let f(x) = cos 3x, let us find a function F(x) such that $\frac{d}{dx}\big(F(x)\big) = \cos 3x$.

We know that $\frac{d}{dx}(\sin 3x) = 3\cos 3x \implies \frac{d}{dx}(\frac{1}{3}\sin 3x) = \cos 3x$.

Here $F(x) = \frac{1}{3}\sin 3x$. In other words we say that the integral $\cos 3x$ is $\frac{1}{3}\sin 3x$.

Let us define integral of a function in general as follows.

Let F(x) be a function such that

$$\frac{d}{dx}[F(x)] = f(x),$$

then ***F(x) is called an integral*** of $f(x)$, with respect to (x). But

$\frac{d}{dx}[F(x) + C] = f(x).$

In general, integral of $f(x)$ is $F(x)$+C, where C is called the constant of integration.

In symbols we write this as $\boldsymbol{\int f(x)dx = F(x) + C}$

List of the standard integrals.

1. $\int x^{n}\, dx = \frac{x^{n+1}}{n+1} + C$
2. $\int \frac{dx}{x} = ln\, |x| + C$
3. $\int e^x dx = e^x + C$
4. $\int a^x dx = \frac{a^x}{ln\, a} + C$
5. $\int sin\, x\, dx = -cos\, x + C$
6. $\int cos\, x\, dx = sin\, x + C$
7. $\int 0dx = c$
8. $\int kdx = kx + C$
9. $\int \ln x\, dx = x \ln x - x + C$
10. $\int \frac{dx}{x \ln x} = \ln|\ln x| + C$
11. $\int \log_b x\, dx = x\frac{\ln x - 1}{\ln b} + C$
12. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C$

13. $\int \frac{-dx}{\sqrt{a^2 - x^2}} = \arccos\frac{x}{a} + C$

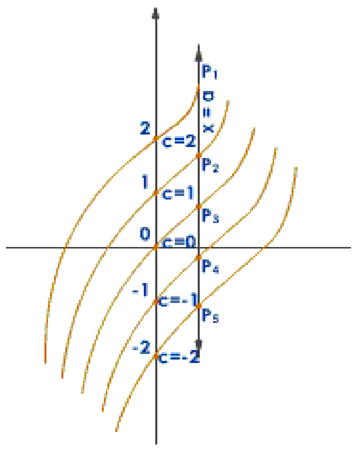
14. $\int tg\, xdx = -\ln|\cos x| + C$

15. $\int ctg\, xdx = ln|\sin x| + C$

1.2. - Indefinite Integral Geometrical Interpretation.

Let $f(x) = 3x^2 \Longrightarrow \int f(x)dx = x^3 + C$

Note that for different values of (C) we get different integrals. But all these integrals are very similar geometrically.



The function $y = x^3 + C$ represent a family of integrals. The above figure shows different curves of the integral function $y = x^3 + C$. These curves fill the co-ordinate plane without overlapping. These curves together constitute the indefinite integrals.

If we draw a line $x = a$ perpendicular to X-axis. Then the curves $y = x^3 + C$ have slopes. The slopes of the tangent at P_1, P_2, P_3, P_4 and P_5 are equal. This indicates, the tangents to these curves are parallel at these points.

1.3. - Indefinite integral properties.

1. Let f(x) be a real value differentiable function, then

$$\boldsymbol{\frac{d}{dx}\int f(x)dx = f(x)} \qquad\qquad \boldsymbol{\int f'(x)dx = f(x) + C}$$

Proof: Let F(x) be any anti- derivative of f(x)

$\frac{d}{dx}[F(x)] = f(x) \rightarrow \int f(x)dx = F(x) + C \rightarrow \frac{d}{dx}[\int f(x)dx] = \frac{d}{dx}[F(x) + C] \rightarrow \frac{d}{dx}[\int f(x)dx] = f(x).$

Similarly, we know that

$f'(x) = \frac{d}{dx}[f(x)] \rightarrow \int f'(x)dx = f(x) + C$, where C is the constant of integration.

2. **Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.**

Proof: Let the two indefinite integrals be: $\int f(x)dx$ and $\int g(x)dx$.

Given: $\frac{d}{dx}\int f(x)\,dx = \frac{d}{dx}\int g(x)dx \rightarrow \frac{d}{dx}[\int f(x)dx - \int g(x)dx] = 0 \rightarrow$
$\int f(x)dx - \int g(x)dx = C$

Where C is any number.

$\int f(x)dx = \int g(x)dx + C_1 \; or \; \int g(x)dx = \int f(x)dx + C_2 \rightarrow$ the family of curves are identical.

3. $\boldsymbol{\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx}$

Proof: By property 1, we have

$\frac{d}{dx}[\int\big(f(x) + g(x)\big)dx = f(x) + g(x)]$

(a)

Also we have

$$\frac{d}{dx}[\int f(x)dx + \int g(x)dx] = \frac{d}{dx}[\int f(x)dx] + \frac{d}{dx}[\int g(x)dx] = f(x) + g(x)$$

(b)

From (a) and (b) we have

$$\frac{d}{dx}\left[\int [f(x) + g(x)]dx\right] = \frac{d}{dx}\left[\int f(x)dx + \int g(x)dx\right] \rightarrow$$

$$\int [f(x) + g(x)] = \int f(x)dx + \int g(x)dx$$

4. For any real number k, $\int kf(x)dx = k\int f(x)dx$

By property 1,we have

$$\frac{d}{dx}\int kf(x)dx = kf(x)$$

(c)

$$\frac{d}{dx}[k\int f(x)dx] = k\frac{d}{dx}[\int f(x)dx] = kf(x)$$

(d)

From (c) and (d) we have

$$\frac{d}{dx}\int kf(x)dx = \frac{d}{dx}[k\int f(x)dx] \rightarrow \int kf(x)dx = k\int f(x)dx$$

Note: *that while using property (2), we can express two equivalent integrals by writing without mentioning constant*

$$\int f(x)dx = \int g(x)dx$$

More generally, combining property (2) and property (3), we can write

$$\int [k_1 f_1(x) + k_2 f_1(x) + k_3 f_3(x) + \cdots + k_n f_n(x)]dx$$
$$= k_1 \int f_1(x)dx + k_2 \int f_2(x)dx + \cdots + k_n \int f_n(x)dx$$

Where f_1, f_2, ….f_n are functions and k_1, k_2, …k_n are real numbers.

Example: Write an anti -derivative of ($\sin 2x$ - $4e^{3x}$) using method of inspection.

$$\frac{d}{dx}(\cos 2x) = -\sin 2x(2) \rightarrow -\frac{1}{2}\frac{d}{dx}(\cos 2x) = \sin 2x \rightarrow \frac{d}{dx}(-\frac{1}{2}\cos 2x) = \sin 2x$$

The anti derivative of (sin2x) is $(-\frac{1}{2} cos\ 2x)$

Similarly $\frac{d}{dx}(e^{3x}) = 3e^{3x} \rightarrow \frac{1}{3}\frac{d}{dx}e^{3x} = e^{3x} \rightarrow 4e^{3x} = \frac{4}{3}\frac{d}{dx}(e^{3x})$

Multiplying both sides by 4 and interchanging RHS and LHS

$$4e^{3x} = \frac{d}{dx}(\frac{4}{3}e^{3x})$$

The anti-derivative of $4e^{3x}$ is $\frac{4}{3}e^{3x}$. The anti-derivative of $\sin 2x - 4e^{3x}$ is $-\frac{1}{2}\cos 2x + \frac{4}{3}e^{3x}$

1.4. - Comparison between differentiation and integration.

1. Both are operations on functions.

2. Both are linear. This is because of the following:

$$\frac{d}{dx}[f_1(x)f_2(x)] = \frac{d}{dx}f_1(x) + \frac{d}{dx}f_2(x)$$

And

$$\int [f_1(x) + f_2(x)]dx = \int f_1(x)dx + f_2(x)dx$$

The constant can be taken outside the differential as well as integral sign as shown below:

$$\frac{d}{dx}[k(x)] = k\frac{d}{dx}f(x)$$

And

$$\int kf(x)dx = k\int f(x)dx$$

3. We have already seen that not all functions are differentiable. Similarly, all functions are not integrable. We will learn about non-differentiable and non-integrable functions in our higher classes.

4. The derivative of a function, when it exists is a unique function. The integral of a function is not so. However, it always differs by a constant only.

5. When a polynomial function P is differentiated, the result is a polynomial whose degree is 1 less than the degree of P. When a polynomial function P is integrated, the result is a polynomial whose degree is 1 more than that of P.

6. We can speak of the derivative of a function at a point. We never speak of the integral of a function at a point, we speak of the integral of a function over an interval on which integral is defined.

7. The derivative of a function has a geometrical meaning, namely, the slope of the tangent to the corresponding curve at the point. Similarly, indefinite integral of a function represents geometrically, a family of curves placed parallel to each other having parallel tangents at the points of intersection of the curves of the family with the lines orthogonal to the axis representing the variable of integration.

8. The derivative is used for finding some physical quantities like the velocity of a moving particle, when the distance traversed at any time t is known. Integral is used to find the distance travelled on time t when velocity at time t is known n.

9. Differentiation is a process involving limits, so is integration.

10. The process of differentiation and integration are inverses of each other. In the earlier section we have found the integral (anti-derivative) of a function by inspection. For a given function f, it may be difficult to find F such that

$\frac{dF}{dx} = f(x)$ or $[\int f(x)dx = F(x)]$

Therefore we need to learn different methods of integration in this section. The three different methods of integration, we learn are

1. Method of substitution

2. Integration using partial fraction.

3. Integration by parts.

1.5. - Integration by Substitution.

Integration of the form

$$\int (g(x))g'(x)dx$$

Let $\int f(x)dx = F(x) + C$

Consider $\int f\big(g(x)\big)g'(x)dx$

Put $g(x) = t$

Differentiating g(x) with respect to t, we have

$$g'(x)dx = dt$$

$$\int f\big(g(x)\big)g'(x)dx = \int f(t)dt = F(t) + C = F\big(g(x)\big) + C$$

Note:

By the above $\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C$

Example:

Integrate the following function $(2x^3+1)^4x^2$

Suggested answer:

If $\quad 2x^3+1=u; du=6x^2dx; x^2dx=\frac{1}{6}du$

$$\int (2x^3+1)^4x^2dx = \frac{1}{6}\int u^4du = \frac{1}{6}\frac{u^5}{5} = \frac{u^5}{30} = \frac{1}{30}(2x^3+1)^5 + C$$

1.6. - Integration using trigonometric identities.

When the integrand consists of trigonometric function, we use suitable trigonometric identities to simplify the function so that it can be integrated. Few identities are given below.

$$sin^2x = \frac{1-\cos 2x}{2}$$

$$cos^2x = \frac{1+\cos 2x}{2}$$

$$sin^3x = \frac{3\sin x - \sin 3x}{4}$$

$$cos^3x = \frac{\cos 3x + 3\cos x}{4}$$

$$\sin A \cos B = \frac{1}{2}[\sin(A-B)+\sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

$$cosA\ cos\ B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

Example:

Integrate the function: $\sin 5x \sin 3x$

Suggested answer:

$$\int \sin 5x \sin 3x\, dx = \frac{1}{2}\int (\cos 2x - \cos 8x)dx = \frac{1}{2}(\frac{1}{2}\sin 2x - \frac{1}{8}\sin 8x) =$$
$$\frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$$

Example:

Integrate the function: $cos^2 x$

Suggested answer:

$$\int cos^2 x dx = \int \frac{1+\cos 2x}{2} dx = \frac{1}{2}\int dx + \int \cos 2x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

1.7. - Integration by Partial fractions.

Before using this technique of integration, let us recall what we have learnt about partial fraction.

Rational function.

If P(x) and Q(x) are two polynomials in x, then the ratio of two polynomials, $\frac{P(x)}{Q(x)}$ is called a rational function, where $Q(x) \neq 0$

Proper rational function

If the degree of the numerator of the rational function is less than that of the denominator, the rational function is called a proper rational function.

$\frac{2x+3}{x^2+5x+7}$ is a proper rational fraction.

Improper rational function.

If the degree of the numerator is greater than the degree of the denominator in a rational fraction, then the rational function is called improper rational function. Like the case of improper fractions reducible to an integer added to a proper fraction, improper rational function can be reduced as a sum of a polynomial and a proper rational function.

In other words, if $\frac{P(x)}{Q(x)}$ is improper rational function, then

$$\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$$

Where T(x) is a polynomial and $\frac{P_1(x)}{Q(x)}$ is a proper rational function.

Partial fractions.

Any proper rational function $\frac{P(x)}{Q(x)}$ can be expressed as sum of rational fractions, each having a factor of Q(x). Each such fraction is known as Partial fraction.

Rule for integrating.

1. Let $\frac{P(x)}{Q(x)}$ be rational function. If $\frac{P(x)}{Q(x)}$ is improper, divide P(x) by Q(x). Let T(x) be the quotient and $P_1(x)$ be the remainder, then

$$\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$$

Where T(x) is a polynomial and $\frac{P_1(x)}{Q(x)}$ is a proper rational function.

2. Resolve the proper rational function $\frac{P_1(x)}{Q(x)}$ in to partial fractions.
3. Write $\frac{P_1(x)}{Q(x)}$ as a sum of partial fractions.
4. Write $\frac{P(x)}{Q(x)}$ as the sum of T(x) and the sum of partial fractions. Integrate each part of the right hand side. This gives the required integral.

Note that if $\frac{P(x)}{Q(x)}$ is a proper rational fraction step 1 need not be performed.

The following table indicates the simpler partial fractions associated to proper rational functions.

№	Form of the rational fraction	Form of the partial fraction
1	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5	$\frac{px^2+qx+r}{(x-a)^3(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \frac{D}{(x-b)}$
6	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{(x-a)} + \frac{Bx+c}{x^2+bx+c}$

In the above table A, B, C and D are real numbers to be determined suitably.

Example:

Integrate the following rational fraction $\int \frac{x^3+x+1}{x^2-1}$

Suggested answer:

Divide the numerator by the denominator, since the rational fraction is improper.

$$\frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1} \qquad (1)$$

Resolve $\frac{2x+1}{x^2-1}$ into partial fraction as follows

$$\frac{2x+1}{x^2-1} = \frac{2x+1}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} \qquad (2)$$

$$\rightarrow \frac{2x+1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)} \rightarrow 2x+1 = A(x+1)+B(x-1)$$

Put x =1, A = 3/2;

Put x = -1, B = ½

Substitution the values of A and B in (2) we have

$$\frac{2x+1}{x^2-1} = \frac{3}{2(x-1)} + \frac{1}{2(x+1)}$$

From (1) we have

$\int \frac{x^3+x+1}{x^2-1} dx = \int \left(x + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}\right) dx = \int x dx + \frac{3}{2}\int \frac{1}{x-1} dx + \frac{1}{2}\int \frac{1}{x+1} dx = \frac{x^2}{2} + \frac{3}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) + C$

1.8. - Integration by parts.

Let u and v be two differentiable function of a single independent variable x. We have

$$\frac{d}{dx}(uv) = \frac{udv}{dx} + \frac{vdu}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - \frac{vdu}{dx}$$

Integrating both sides, we have

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \qquad (1)$$

Let $u = f(x), \frac{dv}{dx} = g(x)$

Then $\frac{du}{dx} = f'(x), v = \int g(x)dx$

(1) Can be written as

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int (\int g(x)dx)f'(x)dx)$$

Note:

$$\boldsymbol{\int f(x)g(x)dx = uv - \int vdu}$$

Note:

1. While integration by parts, the proper choice of first function and second function is significant
2. Integrating by parts may not be applicable to product of functions in all cases. In some cases the product of two functions may not be integrable.
3. While finding the integral of the second function we do not add constant of integration. We need not add a constant of integration to the second function as it gets cancelled in the final results.
4. Sometimes, even if the integral is not a product of two functions, the method of integration by parts can be used.

Example:

Let us integrate $xsin\ x$

Suggested answer:

If $u = x, dv = \sin x\, dx, then\ du = dx, v = \int dv = \int \sin x\, dx = -\cos x$

$$\int x \sin x\, dx = -\,x\cos x + \int \cos x\, dx = -x\cos x + sinx + C$$

Example:

Let us integrate $\frac{\ln x}{x^2}$

Suggested answer:

If $u = \ln x, dv = \frac{dx}{x^2},\ \ then\ \ du = \frac{dx}{x},\ v = \int dv = \int \frac{dx}{x^2} = \int x^{-2} dx = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} + \int \frac{1}{x}\frac{dx}{x} = -\frac{\ln x}{x} + \int x^{-2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C$$

1.9. - Some Special Types of integrals

Following are few special integrals, which can be integrated by using integration by parts

Prove that

$$\int \sqrt{x^2 - a^2}dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Proof:

Let $I = \int \sqrt{x^2 - a^2}dx$

Taking 1 as the second function and integrating by parts, we have

$I = \int \sqrt{x^2 - a^2}\,.(1)dx = \sqrt{x^2 - a^2} \int dx - \iint dx. \frac{1}{2\sqrt{x^2-a^2}}.(2x).dx = x\sqrt{x^2 - a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}.dx = x\sqrt{x^2 - a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}} = x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}dx + \int \frac{a^2}{\sqrt{x^2-a^2}} dx \to I = x\sqrt{x^2 - a^2} - I + a^2 \int \frac{1}{\sqrt{x^2-a^2}} dx \to 2I = x\sqrt{x^2 - a^2} +$

$a^2 log\left|x+\sqrt{x^2-a^2}\right|+C_1 \rightarrow I=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}log\left|x+\sqrt{x^2-a^2}\right|+\frac{C_1}{2}=$ $\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}log\left|x+\sqrt{x^2-a^2}\right|+C$,where $C=\frac{C_1}{2}$

Prove that

$$\int \sqrt{x^2-a^2}dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}log\left|x+\sqrt{x^2-a^2}\right|+C$$

Proof:

Let $I=\int \sqrt{x^2-a^2}\,dx$

Taking 1 as the second function and integrating by parts, we have $I=\int \sqrt{x^2-a^2}\,(1)dx=\sqrt{x^2-a^2}\int 1.dx-\int(\int 1.dx)\frac{1}{2\sqrt{x^2-a^2}}.2x=x\sqrt{x^2-a^2}-\int\frac{2x^2dx}{2\sqrt{x^2-a^2}}=x\sqrt{x^2-a^2}-\int\frac{x^2}{\sqrt{x^2-a^2}}dx=x\sqrt{x^2-a^2}-\int\frac{(x^2+a^2)-a^2}{\sqrt{x^2-a^2}}dx=$ $x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}dx+a^2\int\frac{1}{\sqrt{x^2-a^2}}dx \rightarrow x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}dx+$ $a^2\int\frac{1}{\sqrt{x^2-a^2}}dx \rightarrow I=x\sqrt{x^2-a^2}-I+a^2\int\frac{1}{\sqrt{x^2-a^2}}dx \rightarrow 2I=x\sqrt{x^2-a^2}+$ $a^2 log\left|x+\sqrt{x^2-a^2}\right|+C_1 \rightarrow I=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}log\left|x+\sqrt{x^2-a^2}\right|+\frac{C_1}{2}=$ $\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}log\left|x+\sqrt{x^2-a^2}\right|+C, where\ C=\frac{C_1}{2}$

Integral of the form

$$\int \sqrt{ax^2+bx+C}\,dx$$

Method:

The quadratic expression $ax^2 + bx + c$ can be expressed in the form $a(x^2 \pm A^2)$ by the method of completing the square. The integrals can be evaluated by using the special integrals.

$$\int (px+q)\sqrt{ax^2+bx+C}$$

Let $I = \int px + q)\sqrt{ax^2 + bx + C}$

Put $px + q = L\frac{d}{dx}(ax^2 + bx + c) + M$ (1)

Find the values of the constants L and M by comparing the coefficients of like powers of x on both sides.

Substitute $px + q = L\frac{d}{dx}(ax^2 + bx + C) + M \ \ in\ the\ integral\ I$

The integrals in the form are easily integrable.

Example:

Evaluate the integral $\int \sqrt{3 - 2x - x^2}dx$

Suggested answer:

$$\int \sqrt{3 - 2x - x^2}dx$$

$$= \int \sqrt{3 - (2x + x^2)}dx$$

$$= \int \sqrt{3 - (x^2 + 2x + 1 - 1}dx = \int \sqrt{4 - (x + 1)^2dx}, put\ x + 1 = t$$

$$dx = dt$$

$$= \int \sqrt{2^2 - t^2}dt = \frac{1}{2}t\sqrt{4 - t^2} + \frac{4}{2}sin^{-1}\frac{t}{2} + C$$

$$= \frac{1}{2}(x + 1)\sqrt{3 - 2x - x^2} + 2sin^{-1}\left(\frac{x + 1}{2}\right) + C$$

Summary

$$If\ \frac{d}{dx}\{F(x) + C\} = f(x)\ \ then\ \int f(x)dx = F(x) + C$$

f(x) is called the integrand, F(x) is called the particular integral and C the constant of integration.

$$If \int f(x)dx = F(x) + C \quad then \quad \int f(ax+b)dx = \frac{F(ax+b)}{a} + C$$

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

$$\int Kf(x)dx = K \int f(x)dx, K\ is\ constant$$

1.10. - Method of substitution:

If the integrand f(x) of the integral is not in an integral form the variable of integration x is changed to a suitable variable z by substitution and on differentiation and simplification, the new integral is found integrable.

$$\int f(x)dx = \int f[\phi(z)]\,\phi' dZ \quad when\ x = \phi(z)$$

$$\int f(x)dx = \int \frac{g'(x)}{g(x)} \quad then \quad \int f(x)dx = \log g(x) + C$$

Standard integrals

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} log \left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} log \left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{a^2 - x^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + C$$

Evalution of $\int \frac{1}{ax^2+bx+c} dx$

By method of completing squares $ax^2 + bx + c$ is expressed as $A^2 - X^2$ or $X^2 - A^2$ or $A^2 + x^2$ and the integral reduces to

$$\int \frac{1}{a^2 - x^2} dx \;\; or \;\; \int \frac{1}{a^2 + x^2} dx \;\; or \;\; \int \frac{dx}{x^2 - a^2}$$

which can be evaluated using the standard integrals.

Evalution of $I = \frac{px+q}{ax^2+bx+c} dx$

Method:

$$Step 1. Let\ numerator = L \frac{d}{dx}(Denominator) + M\ and\ determine\ L\ and\ M$$

$$Step 2. I = L \int \frac{{}^{d}/_{dx}(Denominator)}{Denominator} + M \int \frac{1}{Denominator}$$

$$Step 3.\ I = Llog(ax^2 + bx + c) + M \int \frac{1}{ax^2 + bx + c} dx$$

Step 4: The second integral can be evaluated by method completing squares.

Standard integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = log\left|x + \sqrt{x^2 - a^2}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = log\left|x + \sqrt{a^2 + x^2}\right| + C$$

$$Evalution\ of\ \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

By method of completing squares $ax^2 + bx + c$ is expressed as $A^2 + X^2$ or $X^2 - A^2$ and the integral reduces to

$$\int \frac{dx}{\sqrt{a^2 - x^2}} \quad or \quad \int \frac{dx}{\sqrt{a^2 + x^2}} \quad or \quad \int \frac{dx}{\sqrt{x^2 - a^2}}$$

which can be evaluated using the standard integrals.

$$Evalution\ of\ \ I = \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$$

Method:

$$Step 1. Let\ px + q = L \frac{d}{dx}(ax^2 + bx + c) + M\ \ and\ determine\ L\ and\ M$$

$$Step\ 2. I = \int \frac{2ax + b}{\sqrt{ax^2 + bx + c}} dx + M \int \frac{1}{\sqrt{ax^2 + bx + c}} dx$$

$$= 2L\sqrt{ax^2 + bx + c} + M \int \frac{1}{\sqrt{ax^2 + bx + c}} dx$$

Step 3: The second integral can be evaluated by method of completing squares.

$$Integral\ of\ the\ form\ \ \int \frac{1}{asinx + bcosx + c} dx, \frac{dx}{a + bcosx}, \frac{dx}{a + bsinx}$$

Method

$$Step\ 1. Put \tan\frac{x}{2} = t, \sin x = \frac{2t}{1 + t^2}, \cos x = \frac{1 - t^2}{1 + t^2}, dx = \frac{2dt}{1 + t^2}$$

$$Step\ 2. Integral\ \ reduces\ to\ the\ form\ \int \frac{dt}{At^2 + Bt^2 + c}$$

Step 3: Resulting integral is evaluated by method of completing squares.

$$Integral\ of\ the\ form\ \int \frac{a_1 \cos x + b_1 \sin x}{\text{a}\cos x + bsin\ x} dx$$

Method

$Step\ 1. Put\ Numerator = L(Denominator) + M\frac{d}{dx}(Denominator)$

Step 2: Determine L and M

$Step\ 3. Integral\ reduce\ to\ \ I = L\int 1dx + M\int \frac{f'(x)}{f(x)}dx$

Step 4: Integral = Lx + M log (a cos x + bsinx)

Integration by partial fractions:

$Step\ 1. Let\ \frac{f(x)}{g(x)} be\ a\ ration\ function$

$If\ \frac{f(x)}{g(x)}\ is\ not\ proper\ then\ divide\ f(x) by\ g(x) and\ express\ in\ the\ form\ \frac{f(x)}{g(x)}$
$= q(x)$
$+ \frac{r(x)}{g(x)}\ where\ q(x)\ is\ the\ quotient, r(x) is\ theremaider\ and\ \frac{r(x)}{g(x)}\ into\ partial\ fractions$

$Step\ 2. Resolve\ \frac{r(x)}{g(x)}\ into\ partial\ fractions$

$Step\ 3. Write\ \frac{f(x)}{g(x)} = q(x) + sum\ of\ partial\ functions\ of\ \frac{r(x)}{g(x)}$

Step 4: Integrate each part on the right hand side to obtain the required integrals.

Integration by parts

$$\int uvdx = u\int vdx - \int \left(\int vdx\right)\left(\frac{du}{dx}\right)dx$$

In words: Integral of the product of two functions

$$= (1st\ function)(Integral\ of\ 2nd) - \int (Integral\ of\ 2nd)\frac{d}{dx}(1st\ function)dx$$

If the integrand is the product of two functions of different types then their order is determined by the word ILATE where

I = Inverse trigonometric, L = Logarithmic, A = Algebraic, T = Trigonometric, E = Exponential

In the integrand, the first function is the function which comes first in the word ILATE. However, there is no rigid rule in this that you have to select the first function in this order.

Standard integrals

$$\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} sin^{-1}\frac{x}{a} + C$$

$$\int \sqrt{\boldsymbol{x^2 + a^2}}\, dx = \frac{x}{2}\sqrt{\boldsymbol{x^2 + a^2}} - \frac{\boldsymbol{a^2}}{\boldsymbol{2}}\boldsymbol{log}\left\{\boldsymbol{x} + \sqrt{\boldsymbol{x^2 + a^2}}\right\} + \boldsymbol{C}$$

$$\int \sqrt{\boldsymbol{x^2 - a^2}}\, dx = \frac{x}{2}\sqrt{\boldsymbol{x^2 - a^2}} - \frac{\boldsymbol{a^2}}{\boldsymbol{2}}\boldsymbol{log}\left\{\boldsymbol{x} + \sqrt{\boldsymbol{x^2 - a^2}}\right\} + \boldsymbol{C}$$

$Integral\ of\ the\ form\ \int \sqrt{ax^2 + bx + c}\, dx$

Step 1: By method of completing squares

$$\sqrt{ax^2 + bx + c} = \sqrt{A^2 \pm x^2} \quad or\ \sqrt{x^2 - A^2}$$

Step 2: Use standard and integrals and evaluate

$Integral\ of\ the\ form\ I = \int (px + q)\sqrt{ax^2 + bx + c\ dx}$

Step 1: put px + q = L (2ax + b) + M and determine L and M.

$$Step\ 2. I = L\int (2ax + b)\sqrt{ax^2 + bx + c}\ dx + M\int \sqrt{ax^2 + bx + c}\ dx$$

$$= L\frac{2}{3}(ax^2 + bx + c)^{3/2} + M\int \sqrt{ax^2 + bx + c}\ dx$$

Step 3: Second integral on the right hand side can be evaluated by method of completing squares.

2. Definite Integral

Let f be a continuous non-negative function defined on a closed interval [a, b]. Since the value of the function is non- negative, the graph of the function is a curve above X-axis. Let the graph of the curve be as shown in the figure.

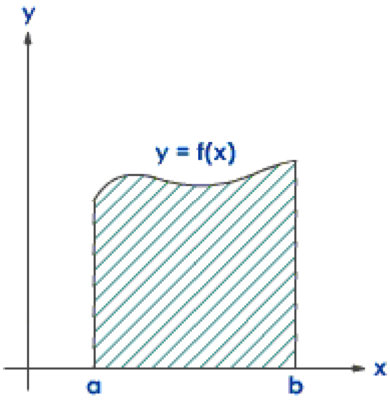


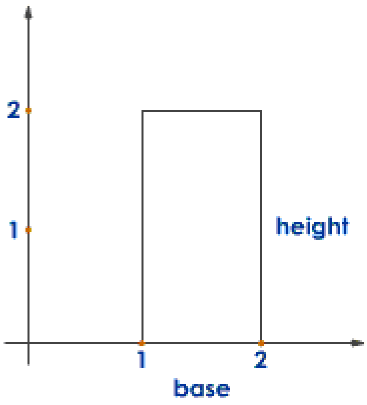
figure (a)

The question is how we find the area under the curve y = f(x) bounded by the X-axis and the lines x = a and x = b. This region is shaded in the graph.

To understand this problem easily let us consider three special such functions.

1. Let $f(x) = 2 \qquad x \in [1,2]$

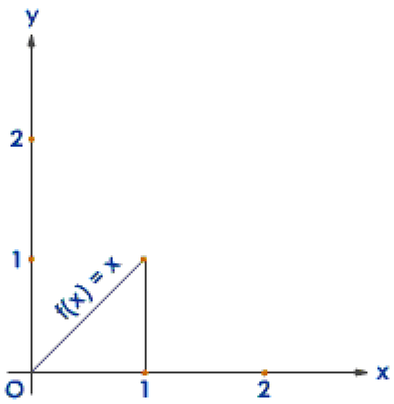
This function is continuous, non-negative in the interval [1, 2], which is shown in the figure.



Being a rectangular region, the area of f(x) = 2 bounded by X- axis, x = 1 and x = 2 is given by base X height, the height being equal to $\frac{f(2)+f(1)}{2}$

Base=(2-1)=1 unit, height =2 unit

2. Consider the function $f(x) = x, x\epsilon[0,1]$

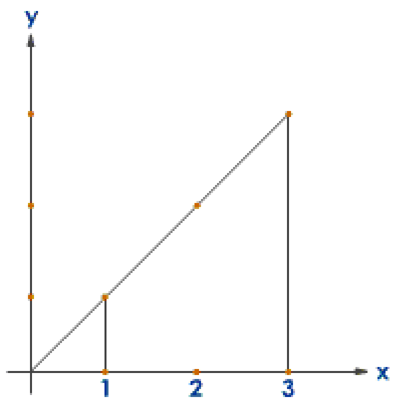


This region is triangular above the axis bounded by x = 0 and x = 1.

The area of this region is given by $\frac{1}{2} \times base \times height$ or $base \times \left(\frac{1}{2} height\right)$

$Area = (1-0) \times \left[\frac{1}{2} \times \left(f(1) + f(0)\right)\right] = \frac{1}{2}$ square units

3. Consider the function $f(x) = x \ \ for \ \ x \in [1,3]$



The region under the centre bounded by X - axis, x = 1 and x = 3 is a trapezium, where area is given by $(3-1)\times\left[\frac{1}{2}(f(3)+f(1))\right]$ (Since the area of the trapezium $=base\times\frac{1}{2}$ (the sum of the parallel sides)).

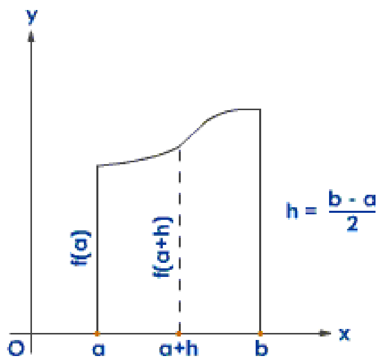
In all the three cases, we have seen that, the area of the regions are obtained by multiplying the base with average height of the curve.1

Using this fact, how can we find the area under the curve in figure (a) above?

The base is the length of the domain interval [a, b] = b - a. Now our problem is to find the average height of the curve. This is indeed the average value of the function in the interval [a, b].

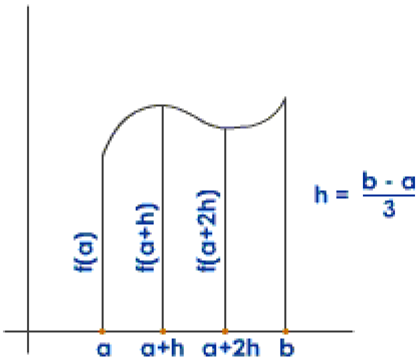
2.1. - Average Value of a Function in an Interval

We can take the value of f at a (i.e., f(a)) as first estimate for average value of the function.



Divide [a, b] into two equal parts such that $h = \frac{b-a}{2}$ then the second estimate of the average value of the function can be taken as second estimate of the average value of the function can be taken as $\frac{f(a)+f(a+h)}{2}$ (see the above figure)

Clearly the second estimate of the average value is better than the first estimate.

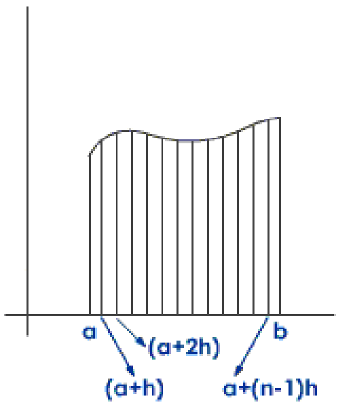


If we divide the interval into three equal parts such that $h = \frac{b-a}{3}$ then the improved estimate for the average value of f(a) is $\frac{f(a)+f(a+h)+f(a+2h)}{3}$ (see the above figure)

In this process, if we divide the closed interval [a, b] into more and more equal parts, and take the average of functional values at these points, we are closer to the average value of the function in closed interval [a, b].

Let us divide the closed intervals to n equal parts, then the average value of the function is $\frac{f(a)+f(a+h)+f(a+2h)+\cdots.+f(a+(n-1)h)}{n}$ (1)

where $h = \frac{b-a}{n}$ as shown in the figure below



For larger value of n, equation (1) will be appropriate estimate for the average value of the function in the given closed interval. With this discussion, we can define average value of f in [a, b]

$$\lim_{n\to\infty} \frac{f(a)+f(a+h)+f(a+2h)+..+f[a+(\overline{n-1})h]}{n}$$

Note that as $n \to \infty, h \to 0,\ nh \to b-a$

Therefore the area under the curve y = f(x) bounded by X-axis, x = a and x = b. = base x average height $= (b-a) \times \lim_{n\to\infty} \frac{f(a)+f(a+h)+f(a+2h)+..+f[a+(\overline{n-1})h]}{n} = \lim_{n\to\infty} \frac{(b-a)}{n} \times [f(a)+f(a+h)+f(a+2h)+..+f(a+(\overline{n-1})h] = \lim_{h\to 0} h[f(a)+f(a+h)+f(a+2h)+..+f(a+(n-1)h)]$

where $nh \to b-a$

2.2. - Definite Integral

Let f (x) be a single valued continuous function defined in the interval [a,b] where b > 0 and let the interval [a,b] be divided into n equal parts each of length h, so that nh = b - a; then we define

$$\int_a^b f(x)dx = \lim h[f(a)+f(a+h)+(a+2h)+\cdots+f(a+(n-1)h)]$$

when $n \to \infty, h \to 0\ and\ nh \to b-a$

Thus, $\int_a^b f(x)dx = \lim_{h\to 0} h\sum_{r=0}^{n-1} f(a+rh)$

where $n \to \infty\ as\ h \to 0$ and remains equal to b-a. We call $\int_a^b f(x)dx$ as the definite integral of f(x) between the limits a and b.

The method of evaluating $\int_a^b f(x)dx$ by using the above definition is called integration from first principles.

2.3. - Definite Integral Through Area of Triangles

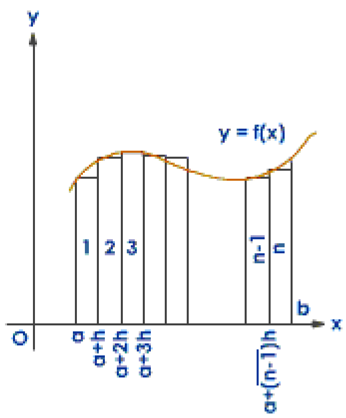
The definition,

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a) + 9a + h) + A + f(a + \overline{n-1}h)] \qquad (2)$$

where $h = \frac{b-a}{n}$ can be explained in another way also. We rewrite above definition as

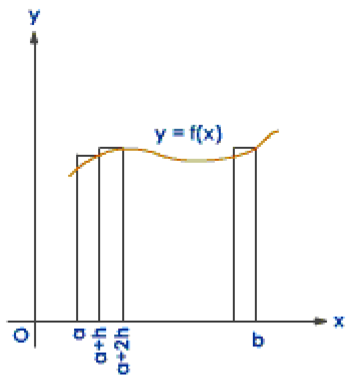
$$\int_a^b f(x)dx = \lim_{n\to\infty}[hf(a) + hf(a+h) + A + hf(a + \overline{n-1}h)], h = \frac{b-a}{n} \qquad (3)$$

Here the first term is hf (a). It is the area of the rectangle marked as 1 in figure below (because h and f (a) are the adjacent sides of this rectangle). Similarly, the second term hf (a+h) is the area of the rectangle marked as 2 in the figure below.



Thus, $[hf(a) + hf(a+h) + \cdots + hf(a + \overline{n-1}h)]$, is the sum of the areas of these n rectangles marked in above figure. The union of these rectangles is approximately the region between the curve and the x-axis. When n is larger, the number of rectangles is more, and the approximation is closer. Therefore if we take the limit as $n \to \infty$ □ we obtain that $\int_a^b f(x)dx$ as in equation (2) is the area of the region bounded by the curve y = f(x) and the lines y = 0, x = a and x = b.

If we take the right end-points instead of the left, then also, we get the same areas as the limit of areas of unions of some other rectangles.



This explains that $\int_a^b f(x)dx$ as in

$$\int_a^b f(x)dx = (b-a)\lim_{n\to\infty}\frac{1}{n}[f(a+h)+f(a+2h)+f(b)]\,, h = \frac{b-a}{n}$$

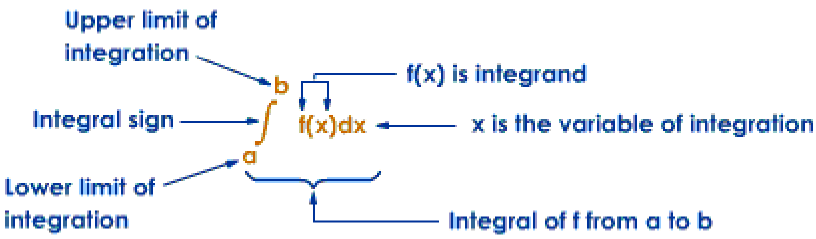
is the area of the same region.

Note that any one of the processes, viz., taking the left hand end-points or the right hand end-points will be sufficient for calculating the desired area.

Terminology

We have the following terminology associated with the symbol

$$\int_a^b f(x)dx$$



Remark

The value of the definite integral of a function over any particular interval depends on the function and the interval, but not on the variable of integration that we choose to represent the independent variable. If the independent variable is denoted by t or u instead of x, we simply write the integral as

$$\int_a^b f(t)dt \quad or \quad \int_a^b f(u)du \quad \text{instead of} \quad \int_a^b f(x)dx$$

Hence, the variable of integration is called a dummy variable.

Example:

Integrate the following definite as limit of sums:

$$\int_0^4 (x + e^{2x})\, dx$$

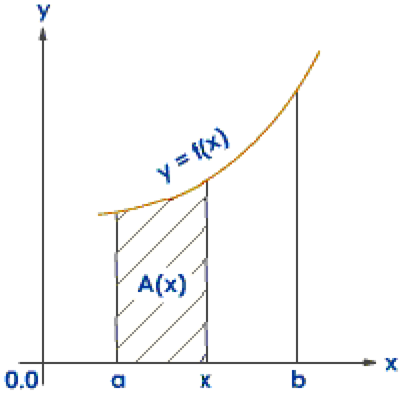
Solution:

We are given that a = 0, b = 4 $\quad h = \frac{4-0}{n} = \frac{4}{n} \quad or\ nh = 4$

By definition $\quad \int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a+h) + \cdots + f(a + \overline{(n-1)}h)]$ $\int_0^4 (x + e^{2x})dx = \lim_{n\to 0} h[f(0) + f(0+h) + f(0+2h) + \cdots + f(0 + (n-1)h)] = \lim_{h\to 0} h\big[(0+e) + (h + e^{2h}) + \left(2h + e^{2(2h)}\right) + \left(3h + e^{3(2h)}\right) + \cdots ((n-1)h + e^{(n-1)2h})\big] = \lim_{h\to 0} h\left[h[1 + 2 + 3 + \cdots (n-1) + (1 + e^{2h} + (e^{2h})^2 + (e^{2h})^3 + \cdots (e^{2h})^{n-1})]\right] = \lim_{h\to 0}\left[\frac{h^2 n(n-1)}{2} + h\left(\frac{1-(e^{2h})^n}{1-e^{2h}}\right)\right] = \lim_{h\to 0}\left[\frac{4^2 - nh^2}{2} - (1-e^8)\frac{1}{2x\frac{1-e^{2h}}{2h}}\right] = \frac{16}{2} - (1-e^8)\lim_{h\to 0}\left(\frac{2h}{1-e^{2h}}\right)\left(\frac{1}{2}\right) = \frac{16}{2} - (1-e^8)\frac{1}{2} = \frac{16}{2} - \frac{1}{1} - \frac{e^8}{2} = \frac{15-e^8}{2}$

2.4. - Area function

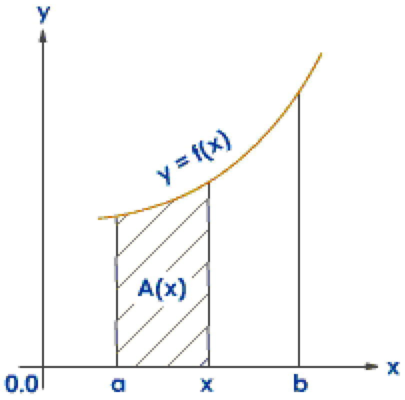
We have already defined, for a continuous function f(x) on a closed interval [a, b] $\int_a^b f(x)dx$ as the area of the region bounded by the curve y = f(x), X-axis and x= a and x = b.



Let $x \in [a, b]$, we defined the area function

$$A(x) = \int_a^x f(x)dx$$

In other words, area of the shaded region is a function of x. The function A(x) is shown in figure below.



This area function A (x) is the anti derivative of f(x). That is f(x) = A'(x). We state fundamental theorems of integral calculus without proof as they are beyond syllabus.

2.5. - First Fundamental Theorem of Integral Calculus

Let f(x) be a continuous function on the closed interval [a, b]. Let the area function A(x) be defined by $A(x) = \int_a^x f(x)dx$ for $x \geq a$ then $A'(x) = f(x)$ for all $x \in [a, b]$

2.6. - Second Fundamental Theorem of Integral Calculus

Let f(x) be a continuous function defined on an interval [a,b].

If $\int f(x)dx = F(x)$ $then$ $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$ is called the definite integral or *f(x)* between the limits a and b. This statement is also known as 'fundamental theorem of calculus'.

Let $\int f(x)dx = F(x) + C$ then $\int_a^b f(x)dx = F(b) - F(a)$

Note: From the above two theorem, we infer the following $\int_a^b f(x)dx =$

(Anti derivative of the function f(x) at b)

- (Anti derivative of the function f(x) at a)

(ii) The fundamental theorem of integral calculus shows a close relationship between differentiation and integration

(iii) These theorems give an alternate method evaluating definite integral, without calculating the limit of a sum.

Example:

Evaluate the definite integral of the following

$$\int_0^{\pi/4} (2sec^2x + x^2 + 2)dx$$

Solution:

$$\int_0^{\pi/4} (2sec^2x + x^2 + 2)dx = 2\int_0^{\pi/4} sec^2xdx + \int_0^{\pi/4} x^2dx + \int_0^{\pi/4} 2dx = 2\,[tanx]_0^{\pi/4}$$

$$+ \left[\frac{x^3}{3}\right]_0^{\pi/4} + [2x]_0^{\pi/4} = 2\left(tan\frac{\pi}{4} - \tan 0\right) + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 - 0 + \left[\frac{\pi}{2} - 0\right]$$

$$= 2 + \frac{x^3}{192} + \frac{\pi}{2}$$

We know that one of the most important method of evaluation of indefinite integral is method of substitution. While using method of substitution to evaluate definite integrals, following steps are involved.

2.7. - Working rule for Evaluating Definite Integral with Suitable Substitution

Suppose we have to evaluate the integral $\int_a^b f(x)dx$ (1) Let $t = g(x)$ is the suitable substitution. Differentiating, we get $dt = g'(x)\,dx$

(2) Now the new variable is t. The upper limit b and the lower limit a are in terms of x. Change these limits to the new variable g (b) and g(a).

(3) Write $\int_a^b f(x)dx = \int_{g(a)}^{g(b)} \frac{f(x)}{g\prime(x)} dt$ and express $\frac{f(x)}{g\prime(x)}$ in terms of t.

(4) Integrate $\frac{f(x)}{g\prime(x)}$ with respect to t.

Find the value of the integral between the new limits g(a) and g(b).

This gives integral of $\int_a^b f(x)dx$

$$1. \int_b^a f(x)dx = -\int_a^b f(x)dx$$

Proof: LHS = F(a) - F(b) = - [F(b) - F(a)] =$-\int_a^b f(x)dx$

$$2. \int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

Proof: LHS = F(b) - F(a) +F(c) - F(b) = F(c) - F(a) = $\int_a^c f(x)dx$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \quad where \;\; a < b < c$$

Proof: RHS = F(b) - F(a) + F(c) - F(b) = F(c) - F(a)= $\int_a^c f(x)dx = LHS$

$$3. \int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Proof: Put a + b - x = t -dx = dt when x = a, t = b x = b, t = a

$$RHS = \int_b^a f(t)(-dt) = -\int_b^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx = LHS$$

$$4. \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Put a - x = t -dx = dt , When x = 0, t = a x = a, t = 0

$$RHS = \int_a^0 f(t)(-dt) = -\int_a^0 f(t)dt = \int_0^a f(t)dt = \int_0^a f(x)dx = LHS$$

$$5. \int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx$$

Proof: $RHS = \int_0^a f(x)dx + \int_0^a f(2a-x)dx = I_1 + I_2$

Let us evaluate I_2 , let 2a - x = t $\rightarrow -dx = dt \;\; or \; -dt = dx$

When x = 0, t = 2a , when x = a, t = a

$$I_2 = \int_{2a}^{a} -f(t)dt = \int_{a}^{2a} f(t)dt \rightarrow \int_{a}^{2a} f(x)dx \text{ (changing the variable t to x)}$$

$$RHS = I_1 + I_2 = \int_0^{a} f(x)dx + \int_a^{2a} f(x)dx = \int_0^{2a} f(x)dx \text{ (by property 2)} = LHS$$

$$6. \int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & f(2a-x) = f(x) \\ 0 & f(2a-x) = -f(x) \end{cases}$$

Proof: $LHS = \int_0^a f(x)dx + \int_a^{2a} f(x)dx \;\; (1)$

Consider $\int_a^{2a} f(x)dx \;\; of \;\; (1)$, when f (x) = f (2a - x)

$$\int_a^{2a} f(x)dx = \int_a^{2a} f(2a-x)dx$$

Put 2a-x = t, -dx = dt, when x = a, t = a, x = 2a, t = 0

$$\int_a^{2aa} f(2a-x)dx = \int_a^{0} f(t)(-dt) = -\int_a^{0} f(t)dt = \int_0^{a} f(t)dt = \int_0^{a} f(x)dx$$

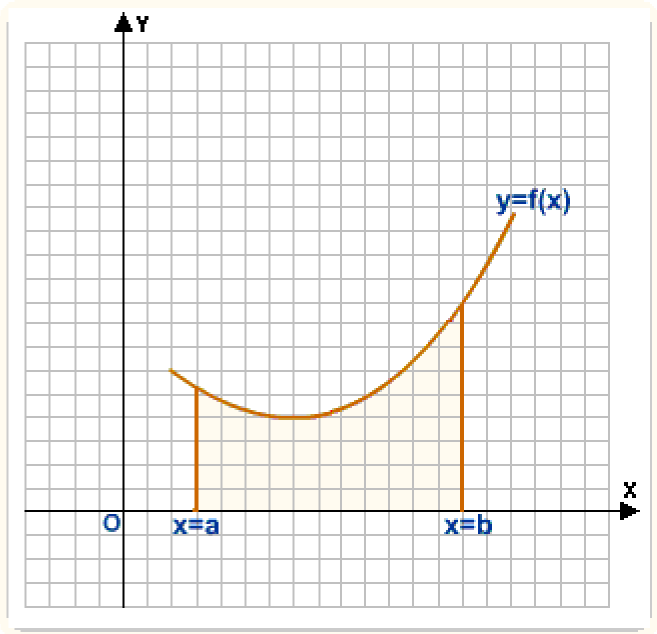
$$LHS = \int_0^{a} f(x)dx + \int_0^{a} f(x)dx = 2\int_0^{a} f(x)dx$$

When f (x) = -f (2a - x), proceeding as above. This value will be equal to

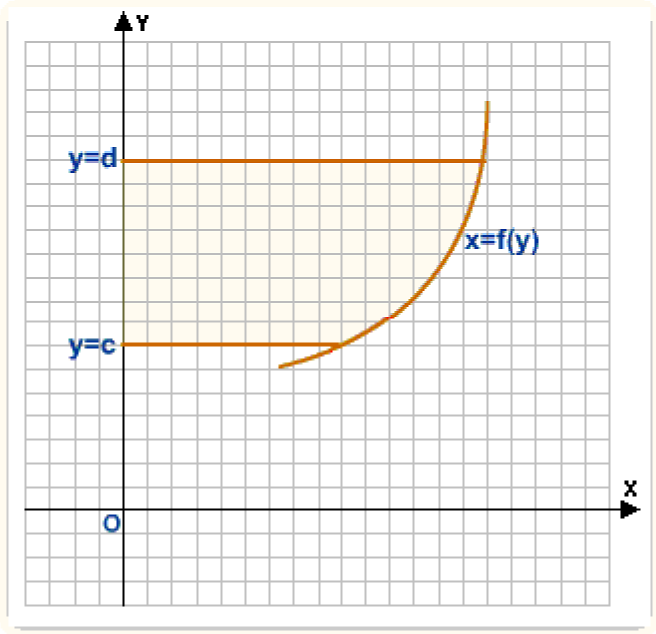
$$\int_{a}^{2a} f(2a-x)dx = -\int_{0}^{a} f(x)dx$$

$$LHS = \int_{0}^{a} f(x)dx - \int_{0}^{a} f(x)dx = 0$$

Let y = f (x) be a curve. The area bounded by y = f (x), x-axis and the ordinates at x = a and x = b is given by $\left|\int_a^b y dx\right| \; or \; \left|\int_a^b f(x)dx\right|$



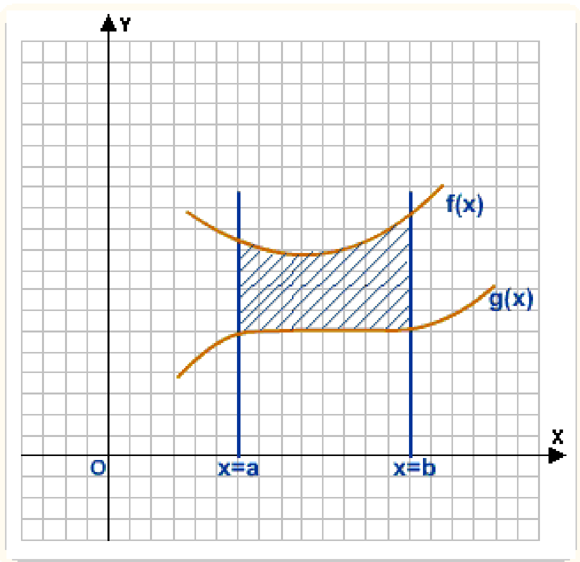
(ii) The area bounded by the curve x = f (y),y = axis and the abscissae at y = c and y = d is given by



$$\left|\int_{c}^{d} x dy\right| \quad or \quad \left|\int_{a}^{b} f(y) dy\right|$$

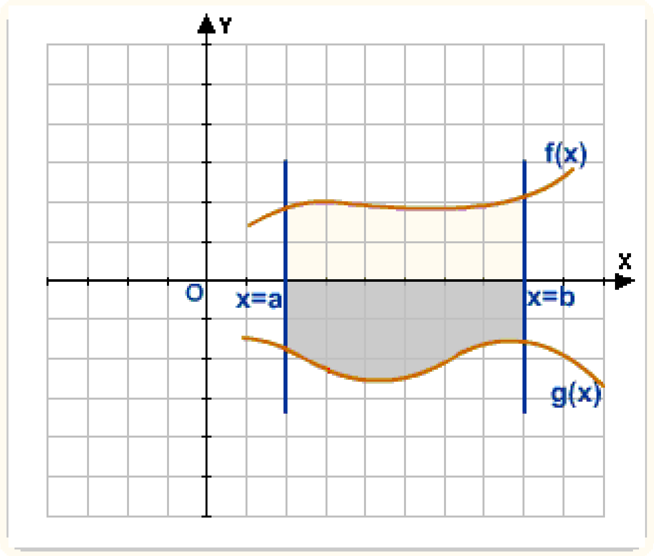
Note 1: The area bounded by the curves f(x) and g(x) and the ordinates x = a and x = b is given by

$$\left|\int_{a}^{b} f(x) dx\right| - \left|\int_{a}^{b} g(x) dx\right|$$



Note 2: If curve f (x) lies above the x-axis and g (x) lies below the x-axis then area bounded by f (x) and g (x) is

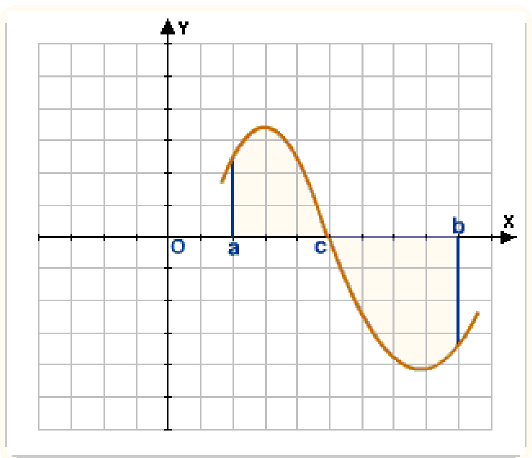
$$\left|\int_a^b f(x)dx\right| + \left|\int_a^b g(x)dx\right|$$



Note 3: If f (a) and f (b) are opposite in sign then the curve crosses the x-axis say at c.

Then the area bounded the curve f (x), x-axis and the ordinates x = a and x = b is

$$\left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$$



3. Double and triple integral

3.1. - Moments and Center of Mass

We have seen in first year calculus that the moments about an axis are defined by the product of the mass times the distance from the axis.

M_x = (Mass)(y) M_y = (Mass)(x)

If we have a region R with density function ρ(x,y), then we do the usual thing. We cut the region into small rectangles for which the density is constant and add up the moments of each of these rectangles. Then take the limit as the rectangle size approaches zero. This will give us the total moment.

3.2. - Definition of Moments of Mass and Center of Mass

Suppose that ρ(x,y) is a continuous density function on a lamina R. Then the moments of mass are

$$M_x = \iint_R \rho(x,y) y dy dx \qquad M_y = \iint_R \rho(x,y) x dy dx$$

and if M is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

Example

Set up the integrals that give the center of mass of the rectangle with vertices (0,0), (1,0), (1,1), and (0,1) and density function proportional to the square of the distance from the origin. Use a calculator or computer to evaluate these integrals.

Solution

The mass is given by

$$M = \int_0^1 \int_0^1 k(x^2 + y^2)dydx = \frac{2k}{3}$$

The moments are given by

$$M_x = \int_0^1 \int_0^1 k(x^2 + y^2)ydydx \qquad and \qquad M_y = \int_0^1 \int_0^1 k(x^2 + y^2)xdydx$$

These evaluate to

$$M_x = \frac{5k}{12} \qquad and \qquad M_y = 5k/12$$

It should not be a surprise that the moments are equal since there is complete symmetry with respect to x and y. Finally, we divide to get

(x,y) = (5/8,5/8)

This tells us that the metal plate will balance perfectly if we place a pin at (5/8,5/8)

3.3. - Moments of Inertia

We often call M_x and M_y the first moments. They have first powers of y and x in their definitions and help find the center of mass. We define the *moments of inertia* (or second moments) by introducing squares of y and x in their definitions. The moments of inertia help us find the kinetic energy in rotational motion. Below is the definition

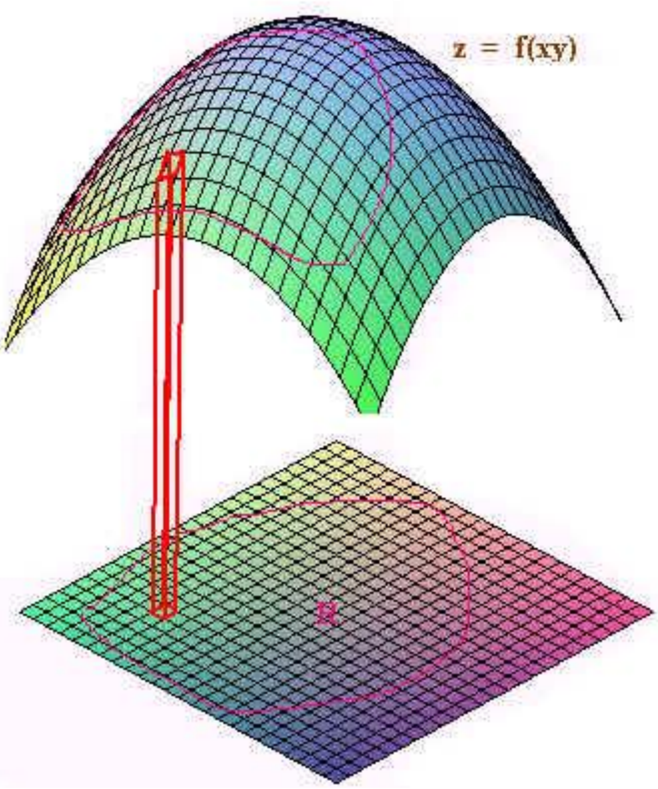
Suppose that ρ(x,y) is a continuous density function on a lamina R. Then the *moments of inertia* are

$$I_x = \iint_R \rho(x,y)y^2dydx \qquad I_y = \iint_R \rho(x,y)x^2dydx$$

3.4. - Surface Area

Definition of Surface Area

We can think of a smooth surface as a quilt flapping in the wind. It consists of many rectangles patched together. More generally and more accurately, let $z = f(x,y)$ be a surface in R^3 defined over a region R in the xy-plane. cut the xy-plane into rectangles. Each rectangle will project vertically to a piece of the surface as shown in the figure below. Although the area of the rectangle in R is



$$\text{Area} = \Delta y \Delta x$$

The area of the corresponding piece of the surface will not be $\Delta y \Delta x$ since it is not a rectangle. Even if we cut finely, we will still not produce a rectangle, but rather will approximately produce a parallelogram. With a little geometry we can see that the two adjacent sides of the parallelogram are (in vector form)

$$\mathbf{u} = \Delta x\, \mathbf{i} + f_x(x,y)\Delta x\, \mathbf{k}$$

and

$$\mathbf{v} = f_y(x,y)\Delta y\, \mathbf{i} + \Delta y\, \mathbf{k}$$

We can see this by realizing that the partial derivatives are the slopes in each direction. If we run Δx in the **i** direction, then we will rise $f_x(x,y)\Delta x$ in the **k** direction so that

rise/run = $f_x(x,y)$

Which agrees with the slope idea of the partial derivative. A similar argument will confirm the equation for the vector **v**. Now that we know the adjacent vectors we recall that the area of a parallelogram is the magnitude of the cross product of the two adjacent vectors. We have

$$|v \times w| = \begin{vmatrix} i & j & k \\ \Delta x & 0 & f_x(x,y)\Delta x \\ 0 & \Delta y & f_y(x,y)\Delta y \end{vmatrix}$$

$$= \left|-\big(f_y(x,y)\Delta y\Delta x\big)i - (f_x(x,y)\Delta y\Delta x)j + (\Delta y\Delta x)k\right|$$

$$= \sqrt{f_y^2(x,y)(\Delta y\Delta x)^2 + f_x^2(x,y)(\Delta y\Delta x)^2 + (\Delta y\Delta x)^2}$$

$$= \sqrt{f_y^2(x,y) + f_x^2(x,y) + 1}\,\Delta y\Delta x$$

This is the area of one of the patches of the quilt. To find the total area of the surface, we add up all the areas and take the limit as the rectangle size approaches zero. This results in a double Riemann sum, that is a double integral. We state the definition below.

Let z = f(x,y) be a differentiable surface defined over a region R. Then its surface area is given by

$$Surface\ Area = \iint_R \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}\,dydx$$

Examples

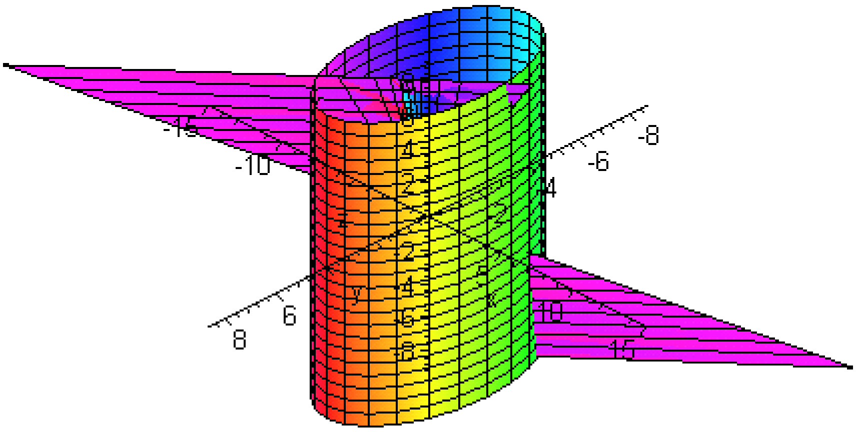
Example

Find the surface area of the part of the plane

$$z = 8x + 4y$$

that lies inside the cylinder

$$x^2 + y^2 = 16$$



Solution

We calculate partial derivatives

$$f_x(x,y) = 8 \qquad f_y(x,y) = 4$$

so that

$$1 + f_x^2(x,y) + f_y^2(x,y) = 1 + 64 + 16 = 81$$

Taking a square root and integrating, we get

$$\iint_R 9\,dy\,dx$$

We could work this integral out, but there is a much easier way. The integral of a constant is just the constant times the area of the region. Since the region is a circle, we get

$$\text{Surface Area} = 9(16\pi) = 144\pi$$

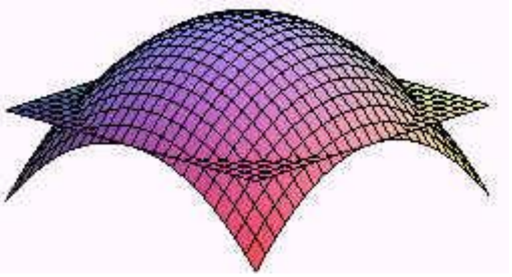
In reality, since there is a square root in the formula, most surface area calculations require intensive integration skills or the use of a machine. The prior example and the next example are not meant to deceive, but rather to show how the essence of surface area problems work without the integration difficulty clouding your understanding.

Example

Find the surface area of the part of the paraboloid

$$z = 25 - x^2 - y^2$$

that lies above the xy-plane.



Solution

We calculate partial derivatives

$$f_x(x,y) = -2x \qquad f_y(x,y) = -2y$$

so that

$$1 + f_x^2(x,y) + f_y^2(x,y) = 1 + 4x^2 + 4y^2$$

At this point if we listen closely, we should hear a little voice pleading "Polar Coordinates". We listen to its call and realize that the region is just the circle

$$r = 5$$

Now convert the integrand to polar coordinates to get

$$\int_0^{2\pi} \int_0^{5} \sqrt{1+4r^2}\, r dr d\theta$$

Now let

$$u = 1 + 4r^2 \qquad du = 8rdr$$

and substitute

$$\frac{1}{8}\int_0^{2\pi} \int_1^{101} u^{1/2} du d\theta = \frac{1}{12}\int_0^{2\pi} u^{3/2}\Big|_1^{101} \approx 530{,}95$$

4. Triple Integrals

4.1 - Definition of the Triple Integral

We have seen that the geometry of a double integral involves cutting the two dimensional region into tiny rectangles, multiplying the areas of the rectangles by the value of the function there, adding the areas up, and taking a limit as the size of the rectangles approaches zero. We have also seen that this is equivalent to finding the double iterated iterated integral. We will now take this idea to the next dimension. Instead of a region in the xy-plane, we will consider a solid in xyz-space. Instead of cutting up the region into rectangles, we will cut up the solid into rectangular solids. And instead of multiplying the function value by the area of the rectangle, we will multiply the function value by the volume of the rectangular solid. We can define the *triple integral* as the limit of the sum of the product of the function times the volume of the rectangular solids. Instead of the double integral being equivalent to the double iterated integral, the triple integral is equivalent to the triple iterated integral.

Let f(x,y,z) be a continuous function of three variables defined over a solid Q. Then the triple integral over Q is defined as

$$\iiint_Q f(x,y,z) = lim \sum f(x,y,z)\Delta x\Delta y\Delta z$$

where the sum is taken over the rectangular solids included in the solid Q and lim is taken to mean the limit as the side lengths of the rectangular solid. This definition is only practical for estimating the triple integral when a data set is given. When we have a symbolically defined function, we use an extension of the fundamental theorem of calculus which is just Fubini's theorem for triple integrals.

4.2. - Theorem for Evaluating Triple Integrals

Let f(x,y,z) be a continuous function over a solid Q defined by

$a < x < b \qquad h_1(x) < y < h_2(x) \qquad g_1(x,y) < z < g_2(x,y)$

Then the triple integral is equal to the triple iterated integral.

$$\iiint_Q f(x,y,z) = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dzdydx$$

Remark: As with double integrals the order of integration can be changed with care.

Example

Evaluate

$\iiint_Q f(x,y,z)dzdydx$

Where

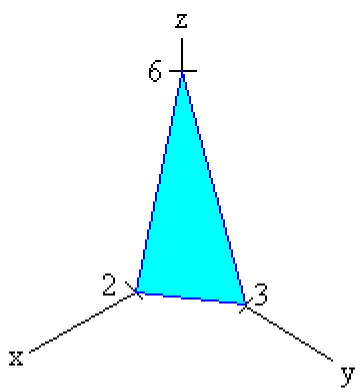
$f(x,y,z) = 1 - x$

and Q is the solid that lies in the first octant and below the plain

$3x + 2y + z = 6$

Solution

The picture of the region



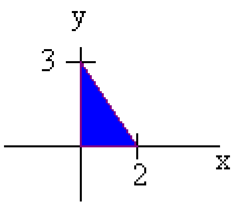
The challenge here is to find the limits. We work on the innermost limit first which corresponds with the variable "z". Think of standing vertically. Your feet will rest on the lower limit and your head will touch the higher limit. The lower limit is the xy-plane or

$z = 0$

The upper limit is the given plane. Solving for z, we get

$z = 6 - 3x - 2y$

Now we work on the middle limits that correspond to the variable "y". We look at the projection of the surface in the xy-plane. It is shown below.



Now we find the limits just as we found the limits of double integrals. The lower limit is just

y = 0

If we set z = 0 and solve for y, we get for the upper limit

y = 3 - 3/2 x

Next we find the outer limits, corresponding to the variable "x". The lowest x gets is 0 and highest x gets is 2. Hence

0 < x < 2

The integral is thus

$$\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} (1-x)\,dz\,dy\,dx = \int_0^2 \int_0^{3-\frac{3}{2}x} [z - xz]_0^{6-3x-2y}\,dy\,dx =$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} [(6-3x-2y) - (6x - 3x^2 - 2xy)]\,dy\,dx =$$

$$= \int_0^2 [\,6y - 9xy - y^2 + 3x^2y + xy^2]_0^{3-\frac{3}{2}x}\,dx =$$

$$= \int_0^2 \Big(18 - 9x - 27x + \frac{27}{2}x^2 - 9 + 9x - \frac{9}{4}x^2 + 9x^2 - \frac{9}{2}x^3 + 9x$$

$$- 9x^2 + \frac{9}{4}x^3\Big)\,dx =$$

$$= \int_0^2 \left(9 - 18x + \frac{45}{4}x^2 - \frac{9}{4}x^3\right) dx =$$

$$= \left[9x - 9x^2 + \frac{15}{4}x^3 - \frac{9}{16}x^4\right]_0^2 = 3$$

Example

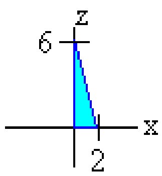
Switch the order of integration from the previous example so that dydxdz appears.

Solution

This time we work on the "y" variable first. The lower limit for the y-variable is 0. For the upper limit, we solve for y in the plane to get

$y = 3 - 3/2\, x - 1/2\, z$

To find the "x" limits, we project onto the xz-plane as shown below



The lower limit for x is 0. To find the upper limit we set y = 0 and solve for x to get

$x = 2 - 1/3\, z$

Finally, to get the limits for z, we see that the smallest z will get is 0 and the largest z will get is 6. We get

$0 < z < 6$

We can write

$$\int_0^6 \int_0^{2-\frac{1}{3}z} \int_0^{6-\frac{3}{2}x-\frac{1}{3}z} (1-x)\,dy\,dx\,dz$$

4.3. - Mass, Center of Mass, and Moments of Inertia

For a three dimensional solid with constant density, the mass is the density times the volume. If the density is not constant but rather a continuous function of x, y, and x, then we can cut the solid into very small rectangular solids so that on each rectangular solid the density is approximately constant. The volume of the rectangle is

ΔMass = (Density)(ΔVolume) = f(x,y,z) $\Delta x\Delta y\Delta z$

Now do the usual thing. We add up all the small masses and take the limit as the rectangular solids get small. This will give us the triple integral

$$Mass = \iiint_Q f(x,y,z)\,dz\,dy\,dx$$

We are often interested in the center of mass of a solid. For example when the NEAR satellite orbited around the asteroid Eros, NASA scientists needed to compute the center of mass of the asteroid. Kepler told us that a stable orbit will always orbit in an elliptical orbit with the center of mass as one of the foci.

The NEAR satellite orbiting around Eros

We find the center of mass of a solid just as we found the center of mass of a lamina. Since we are in three dimensions, instead of the moments about the axes, we find the moments about the coordinate planes. We state the definitions from physics below.

Definition: Moments and Center of Mass

Let ρ(x,y,z) be the density of a solid Q. Then the *first moments* about the coordinate planes are

$$M_{yz} = \iiint_Q x\rho(x,y,z)dzdydx$$

$$M_{xz} = \iiint_Q y\rho(x,y,z)dzdydx$$

$$M_{xy} = \iiint_Q z\rho(x,y,z)dzdydx$$

and the ***center of mass*** is given by

$$(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M},\frac{M_{xz}}{M},\frac{M_{xy}}{M}\right)$$

Notice that letting the density function being identically equal to 1 gives the volume

$$Volume = \iiint_Q dzdydx$$

Exercise

Find the center of mass of the solid that lies below the paraboloid

$z = 4 - x^2 - y^2$

that lies above the xy-plane if the density of the region is given by

$\rho(x,y,z) = x^2 + 2y^2 + z$

You may use your calculator or computer to evaluate the integrals.

Just as with lamina, there are formulas for moments of inertial about the three axes. They involve multiplying the density function by the square of the distance from the axes. We have

Definition: Moments of Inertia

Let ρ(x,y,z) be the density of a solid Q. Then the *first moments of inertia* about the coordinate axes are

$$I_x = \iiint_Q (y^2 + z^2)\rho(x,y,z)dzdydx$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x,y,z)dzdydx$$

$$I_z = \iiint_Q (x^2 + y^2)\rho(x,y,z)dzdydx$$

The Problem

Let

$f(x) = ax^2 + bx + c$

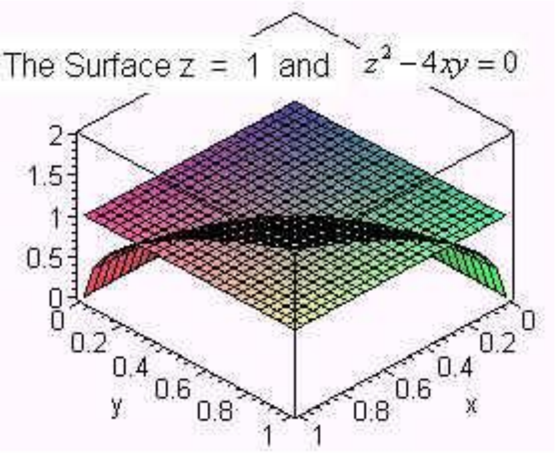
If a,b, and c are chosen randomly from the interval [0,1], what is the probability that f has real roots?

Solution

This is equivalent to finding the volume of the solid that lies inside the unit cube that lies above the discriminate surface

$$z^2 - 4xy = 0$$

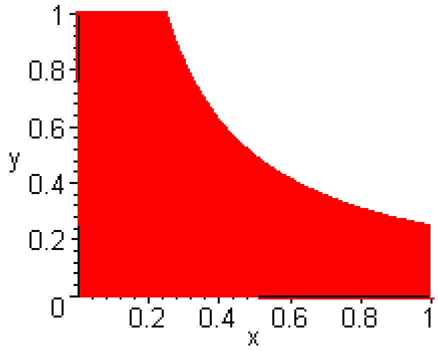
(Here z is b, x is a, and y is c.)



The thing to notice is that the outer limits of the triple integral is not the unit square since the surface rises above $z = 1$ for part of the square. This mistake will lead to the answer of 1/9. Instead it is the part of the unit square that does not lie above the curve

$$4xy = 1$$

which is shown below



We will need to break this up into two integrals as follows

$$\int_0^{1/4}\int_0^1\int_{\sqrt{xy}}^1 dzdydx + \int_{1/4}^1\int_0^{1/4x}\int_{2\sqrt{xy}}^1 dzdydx = \frac{5}{36} + \frac{\ln 2}{6}$$

The solution is approximately equal to .25 which is significantly greater than 1/9.

4.4. - Triple Integrals in Cylindrical and Spherical Coordinates

4.4.1. - Cylindrical Coordinates

When we were working with double integrals, we saw that it was often easier to convert to polar coordinates. For triple integrals we have been introduced to three coordinate systems. The rectangular coordinate system (x,y,z) is the system that we are used to. The other two systems, cylindrical coordinates (r,θ,z) and spherical coordinates (ρ,θ,φ) are the topic of this discussion.

Recall that cylindrical coordinates are most appropriate when the expression

$$x^2 + y^2$$

occurs. The construction is just an extension of polar coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Since triple integration can be looked at as iterated integration we have

$$\int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dzdydx = \int_a^b \int_{h_1(x)}^{h_2(x)} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz \right] dydx =$$

$$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \left[\int_{g_1(r\cos\theta, r\sin\theta)}^{g_2(r\cos\theta, r\sin\theta)} f(rcos\,\theta, r\sin\theta, z)dz \right] rd\, rd\theta$$

$$= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r\cos\theta, r\sin\theta)}^{g_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) rdz\, dr\, d\theta$$

This leads us the the following theorem

Theorem: Integration With Cylindrical Coordinates

Let f(x,y,z) be a continuous function on a solid Q. Then

$$\iiint_Q f(x,y,z)dzdydx = \iiint_Q f(r\cos\theta, r\sin\theta, z) rdzdrd\theta$$

Example

Find the moment of inertia about the z-axis of the solid that lies below the paraboloid

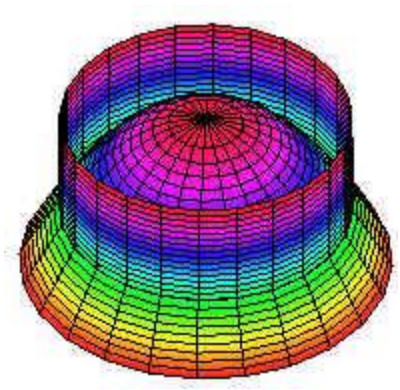
$z = 25 - x^2 - y^2$

inside the cylinder

$x^2 + y^2 = 4$

above the xy-plane, and has density function

$\rho(x,y,z) = x^2 + y^2 + 6z$



Solution

By the moment of inertia formula, we have

$$I_z = \iiint\limits_Q (x^2 + y^2)(x^2 + y^2 + 2z)\,dzdydx$$

The region, being inside of a cylinder is ripe for cylindrical coordinates. We get

$$I_z = \int_0^{2\pi}\int_0^{2}\int_0^{25-r^2} r^2(r^2 + 6z)r\,dzdrd\theta = \int_0^{2\pi}\int_0^{2} [r^5 z + 3r^3 z^2]_0^{25-r^2}\,drd\theta$$

$$\int_0^{2\pi}\int_0^{2} (-125r^5 + 2r^7 + 1875r^3)\,drd\theta = \frac{37384\pi}{3}$$

4.4.2. - Spherical Coordinates

Another coordinate system that often comes into use is the spherical coordinate system. To review, the transformations are

x = ρ cosθ sinϕ y = ρ sinθ sinθ z = ρcosθ

In the next section we will show that

dzdydx = ρ^2 sinϕ dρdϕdθ

This leads us to

Theorem: Integration With Spherical Coordinates

Let f(x,y,z) be a continuous function on a solid Q. Then

$$\iiint_Q f(x,y,z)dzdydx = \iiint_Q f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi d\rho d\phi d\theta$$

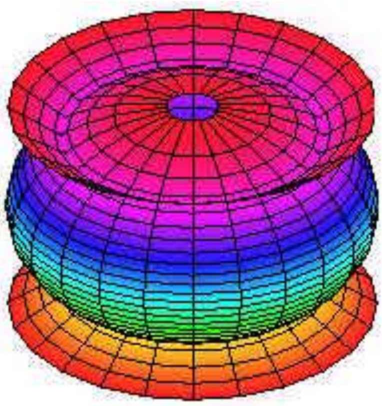
Example

Find the volume of solid that lies inside the sphere

$$x^2 + y^2 + z^2 = 2$$

and outside of the cone

$$z^2 = x^2 + y^2$$



Solution

We convert to spherical coordinates. The sphere becomes

$$\rho = \sqrt{2}$$

To convert the cone, we add z^2 to both sides of the equation

$$2z^2 = x^2 + y^2 + z^2$$

Now convert to

$$2\rho^2\cos^2\phi = \rho^2$$

Canceling the ρ^2 and solving for ϕ we get

$$\phi = cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \text{ or } \frac{7\pi}{4}$$

In spherical coordinates (since the coordinates are □ periodic)

$$7\pi/4 = 3\pi/4$$

To find the volume we compute

$$V = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho d\phi d\theta$$

Evaluating this integral should be routine at this point and is equal to

$$V = \frac{8\pi}{3}$$

4.5. -Jacobians

4.5.1. - Review of the Idea of Substitution

Consider the integral

$$\int_0^2 x\cos(x^2)dx$$

To evaluate this integral we use the u-substitution

$$u = x^2$$

This substitution sends the interval [0,2] onto the interval [0,4]. We can see that there is stretching of the interval. The stretching is not uniform. In fact, the first part [0,0.5] is actually contracted. This is the reason why we need to find du.

$\frac{du}{dx} = 2x$ or $\frac{dx}{du} = \frac{1}{2x}$

This is the factor that needs to be multiplied in when we perform the substitution. Notice for small positive values of x, this factor is greater than 1 and for large values of x, the factor is smaller than 1. This is how the stretching and contracting is accounted for.

4.5.2. - Jacobians

We have seen that when we convert to polar coordinates, we use

$$dydx = r\,dr\,d\theta$$

With a geometrical argument, we showed why the "extra r" is included. Taking the analogy from the one variable case, the transformation to polar coordinates produces stretching and contracting. The "extra r" takes care of this stretching and contracting. The goal for this section is to be able to find the "extra factor" for a more general transformation. We call this "extra factor" the ***Jacobian*** of the transformation. We can find it by taking the determinant of the two by two matrix of partial derivatives.

4.5.3. - Definition of the Jacobian

Let $x = g(u,v)$ and $y = h(u,v)$ be a transformation of the plane. Then the Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/_{\partial u} & \partial x/_{\partial v} \\ \partial y/_{\partial u} & \partial y/_{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example

Find the Jacobian of the polar coordinates transformation

$x(r,\theta) = r\cos\theta \qquad y(r,\theta) = r\sin\theta$

Solution

We have

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\,cos^2\theta + rsin^2\theta = r$$

This is comforting since it agrees with the extra factor in integration.

4.6. - Double Integration and the Jacobian

Theorem: Integration and Coordinate Transformations

Let $T\colon R^2 \to R^2$ given by $x = g(u,v)$, $y = h(u,v)$ be a transformation on the plane that is one to one from a region S to a region R. If g and h have continuous partial derivatives such that the Jacobian is never zero, then

$$\iint_R f(x,y)dydx = \iint_S f(g(u,v),h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

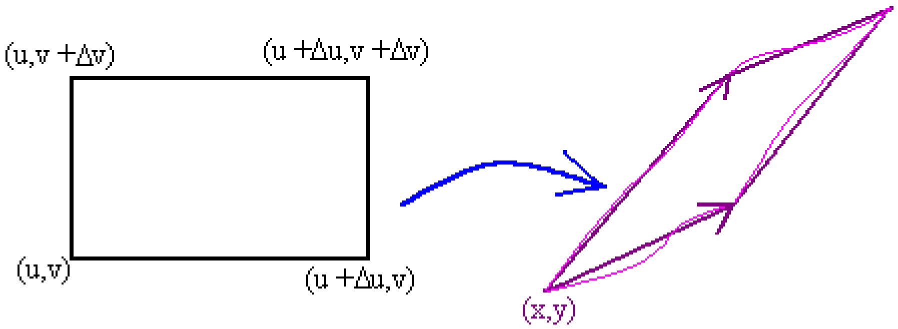
Remark: A useful fact is that the Jacobian of the inverse transformation is the reciprocal of the Jacobian of the original transformation.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = {}^{1}\Big/\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$

This is a consequence of the fact that the determinant of the inverse of a matrix A is the reciprocal of the determinant of A.

Idea of the Proof

As usual, we cut S up into tiny rectangles so that the image under T of each rectangle is a parallelogram.



We need to find the area of the parallelogram. Considering differentials, we have

$T(u + \Delta u,v) \quad T(u,v) + (x_u\Delta u,y_u\Delta u)$

$T(u,v + \Delta v) \quad T(u,v) + (x_v\Delta v,y_v\Delta v)$

Thus the two vectors that make the parallelogram are

$\mathbf{P} = g_u\Delta u\ \mathbf{i} + h_u\Delta u\ \mathbf{j}$

$\mathbf{Q} = g_v\Delta v\ \mathbf{i} + h_v\Delta v\ \mathbf{j}$

To find the area of this parallelogram we just cross the two vectors.

$$P \times Q = \begin{vmatrix} i & j & k \\ x_u\Delta u & y_u\Delta u & 0 \\ x_v\Delta v & y_v\Delta v & 0 \end{vmatrix} = |x_u y_v - x_v y_u|\Delta u\Delta v = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v$$

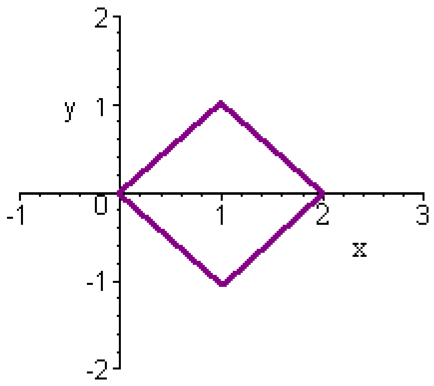
and the extra factor is revealed.

Example

Use an appropriate change of variables to find the volume of the region below

$$z = (x - y)^2$$

above the x-axis, over the parallelogram with vertices (0,0), (1,1), (2,0), and (1,-1)



Solution

We find the equations of the four lines that make the parallelogram to be

$$y = x \qquad y = x - 2 \qquad y = -x \qquad y = -x + 2$$

or

$$x - y = 0 \qquad x - y = 2 \qquad x + y = 0 \qquad x + y = 2$$

The region is given by

$$0 < x - y < 2 \qquad \text{and} \qquad 0 < x + y < 2$$

This leads us to the inverse transformation

$$u(x,y) = x - y \qquad v(x,y) = x + y$$

The Jacobian of the inverse transformation is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

Since the Jacobian is the reciprocal of the inverse Jacobian we get

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$$

The region is given by

0 < u < 2 and 0 < v < 2

and the function is given by

$z = u^2$

Putting this all together, we get the double integral

$$\int_0^2 \int_0^2 u^2 \left(\frac{1}{2}\right) du dv = \int_0^2 \left[\frac{u^3}{6}\right]_0^2 dv$$

$$\int_0^2 \frac{4}{3} dv = \frac{8}{3}$$

4.7. - Jacobians and Triple Integrals

For transformations from R^3 to R^3, we define the Jacobian in a similar way

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Example

Find the Jacobian for the spherical coordinate transformation

$x = \rho\cos\theta \sin\phi$ $y = \rho \sin\theta \sin\phi$ $z = \rho \cos\phi$

Solution

We take partial derivatives and compute

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi & \rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} =$$

$$= |\cos\theta\sin\phi(-\rho^2\cos\theta sin^2\phi) + \rho sin\theta sin\phi(-\rho\sin\theta sin^2\phi - \rho sin\theta cos^2\phi) + \rho\cos\theta cos\phi(-\rho cos\theta cos\phi sin\phi)| =$$

$$= |-\rho^2 cos^2\theta sin^3\phi - \rho^2 sin^2\theta\sin\phi(sin^2\phi + cos^2\phi) - \rho^2 cos^2\theta cos^2\phi\sin\phi|$$

$$= |\rho^2 sin\phi(cos^2\theta sin^2\phi + sin^2\theta + cos^2\theta cos^2\phi)| =$$

$$= |\rho^2 sin\phi(cos^2\theta(sin^2\phi + cos^2\phi) + sin^2\theta)| = |\rho^2\sin\phi(cos^2\theta + sin^2\theta)| = |\rho^2\sin\phi|$$